Cosmology 001

How to build spaces

Euclidean space

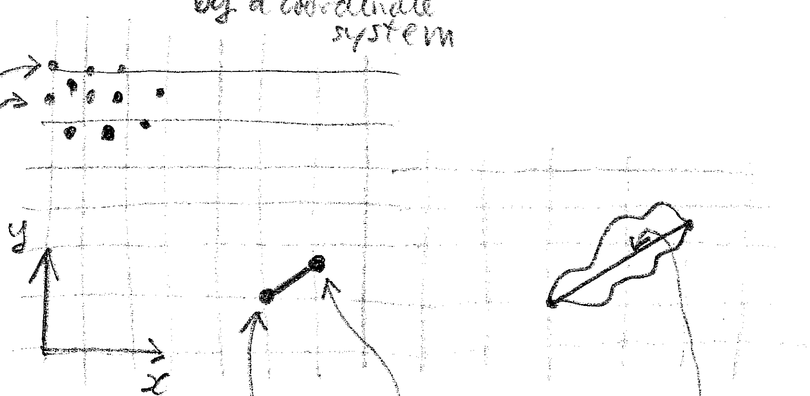
Manifold of points

This means the points can be covered smoothly by a coordinate system

+

metrical structure

Tells us the distance between points. Encoded in line element



Straight lines are geodesics = curves of shortest distance.

This is the straight line!

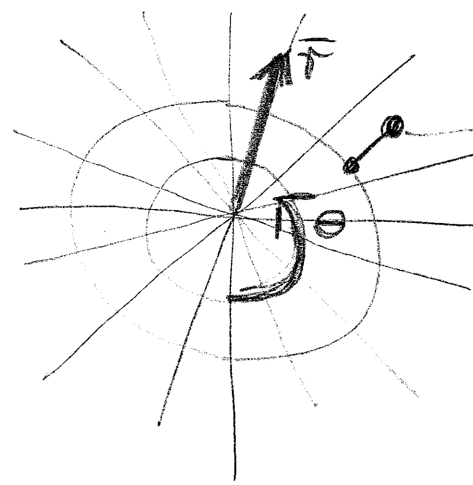
(x,y) $(x+dx, y+dy)$

separated by distance dl where

$$dl^2 = dx^2 + dy^2 \quad \text{[Pythagoras!]}$$

We can use any coordinate system we like.

e.g. radial coordinates



$$dl^2 = d\bar{r}^2 + \bar{r}^2 d\theta^2$$

Define new coordinate

$$r = \bar{r}/R$$

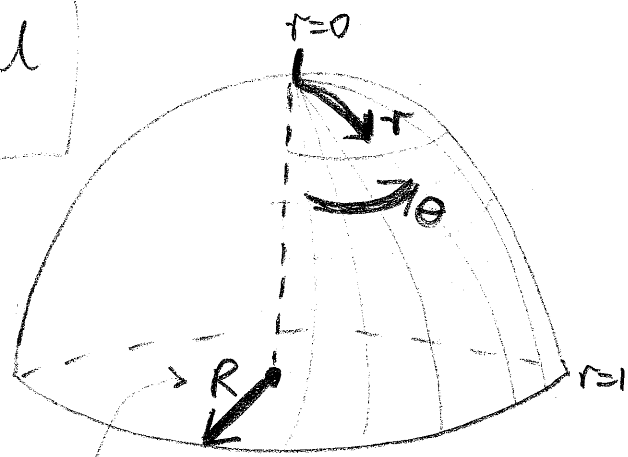
select this value arbitrarily

$$\Downarrow$$

$$dl^2 = R^2\left[dr^2 + r^2 d\theta^2\right]$$

Spaces of Constant curvature

Spherical space



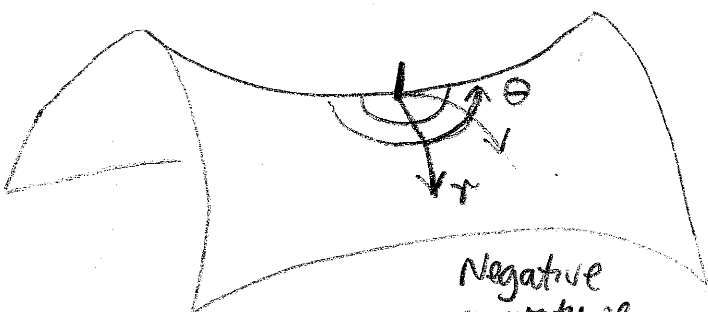
Radius of curvature

Positive curvature

$$dl^2 = R^2 \left[\frac{dr^2}{1-r^2} + r^2 d\theta^2 \right]$$

Hyperbolic space

$$dl^2 = R^2 \left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 \right]$$



Negative curvature

General formula for line element

$$dl^2 = R^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 \right]$$

$k = 0$ Euclidean

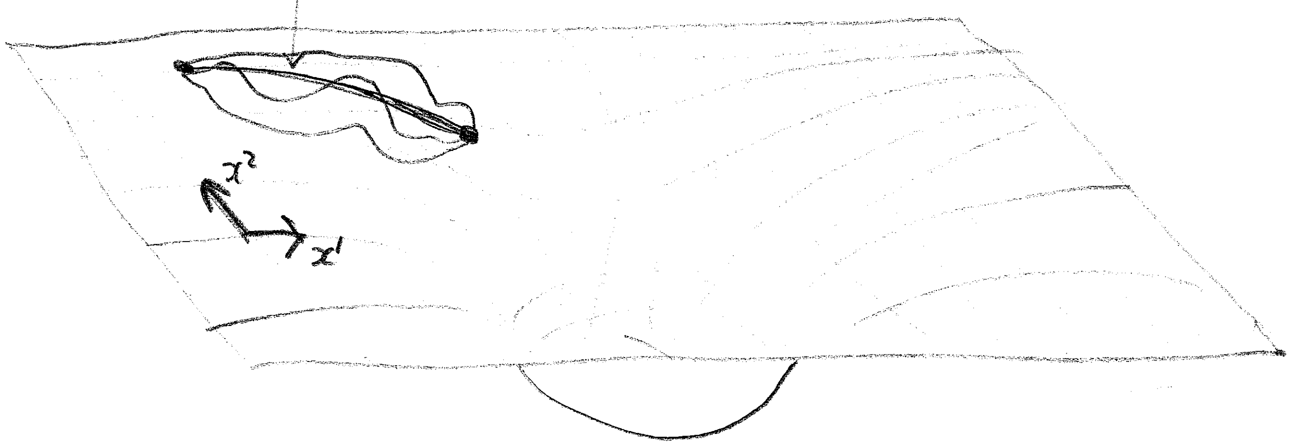
$k = +1$ spherical

$k = -1$ Hyperbolic

"Straight" lines are still geodesics = curves of shortest distance

Spaces of variable curvature

"straight" lines are still geodesics = curves of shortest distance



$$dl^2 = g_{11}(dx^1)^2 + g_{12}\,dx^1dx^2 + g_{21}\,dx^2dx^1 + g_{22}(dx^2)^2$$

$g_{12} = g_{21}$

variability of curvature encoded in the way that the $g_{11}, \dots, g_{22}$ vary from point to point

$$= \sum_{i,k=1,2} g_{ik}\,dx^i dx^k$$

Einstein summation convention

$$= \quad g_{ik}\,dx^i dx^k$$

The matrix of coefficients

$$g_{ik} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

is the metric tensor g in the coordinate system x^1, x^2

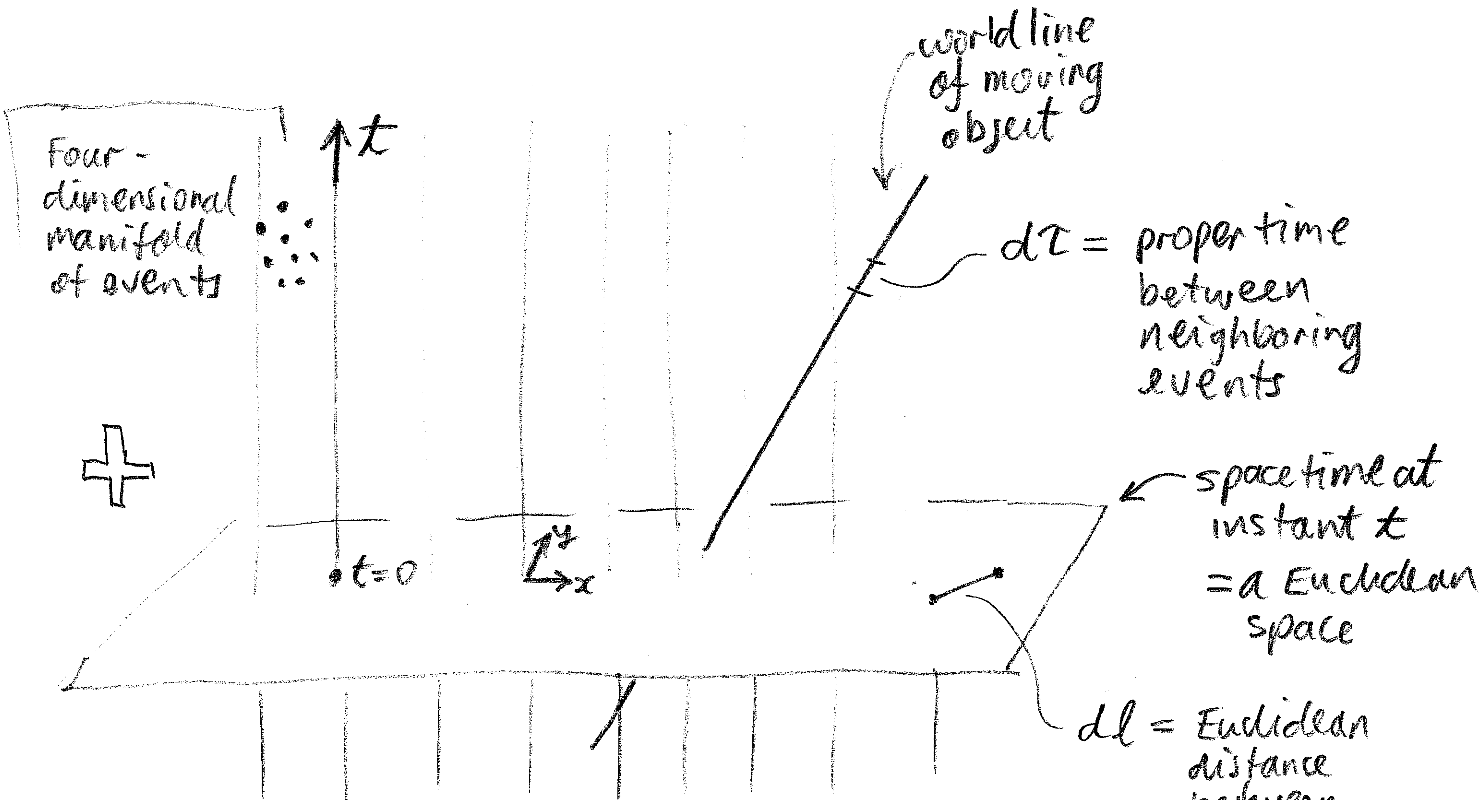
Example: metric for spherical space

$$g_{ik} = \begin{bmatrix} \frac{R^2}{1-r^2} & 0 \\ 0 & R^2r^2 \end{bmatrix}$$

Spacetime

Minkowski spacetime = special relativity

the case of a flat spacetime



Minkowski metric

For neighboring events (x, y, t), $(x+dx, y+dy, z+dz)$

$d\tau, dl$ given via the interval ds

$\boxed{c=1}$

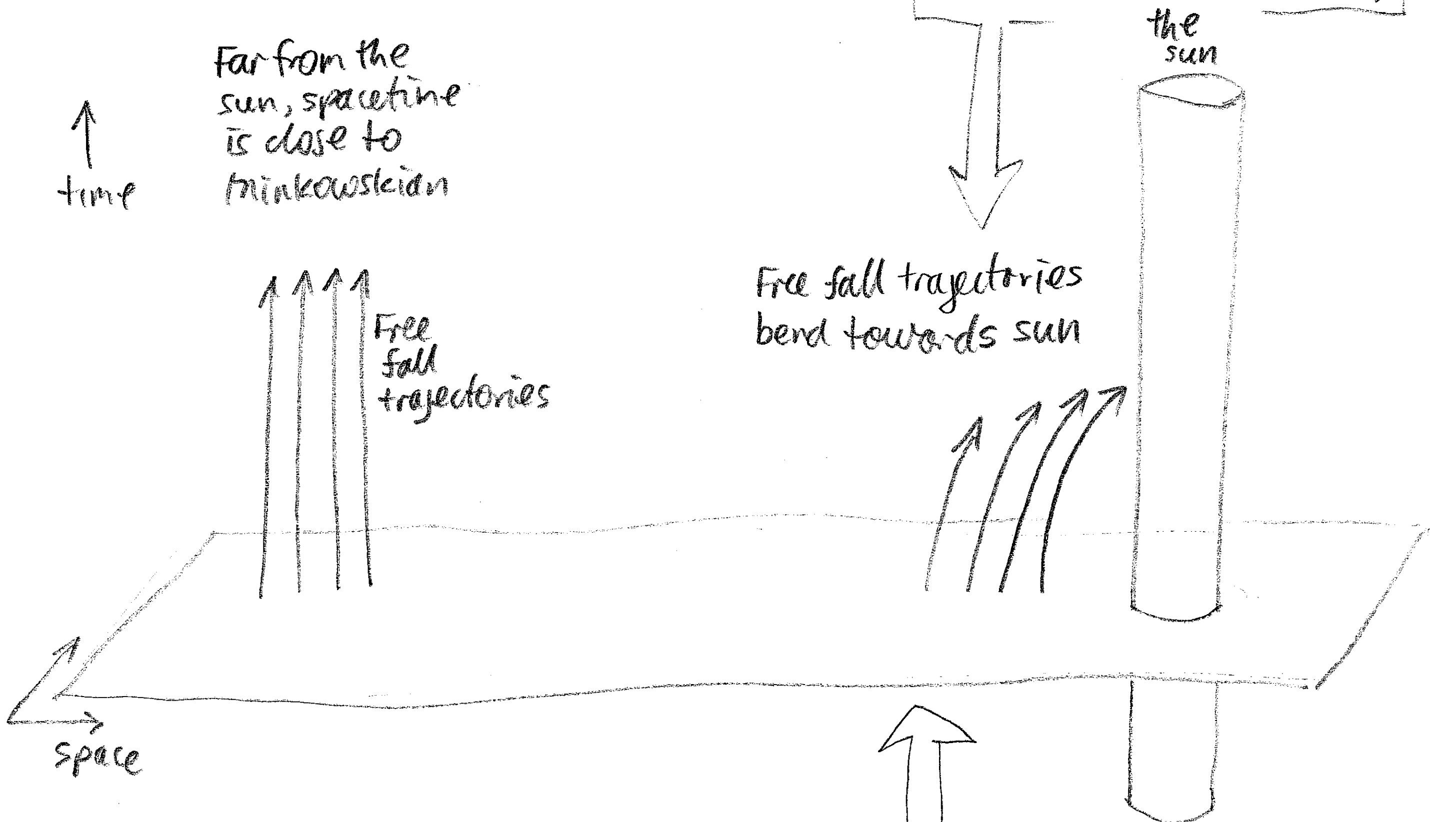
$$ds^2 = dt^2 - (dx^2 + dy^2) \begin{cases} ds = d\tau & \text{timelike intervals} \\ ds = -dl & \text{spacelike intervals} \end{cases}$$

Minkowski's big discovery!

Inertial trajectories = Free fall trajectories = Geodesics = curves of extremal interval ← in this case greatest

General Relativity: The core idea

old view: Gravitational force deflects bodies into sun (Newton)



New view: spacetime near sun is (metrically) curved. Bodies in free fall still follow geodesics, but these geodesics curve towards sun. Farewell to gravitational force

Einstein

Einstein Gravitational Field Equations

Fix metric in $ds^2 = \underbrace{g_{\mu\nu}}_{\text{Represents time, spatial geometry and gravitation}} dx^\mu dx^\nu$ spacetime coordinates $x^1, \dots, x^4$

Spacetime curvature $\Longleftrightarrow$ Energy-momentum density

Einstein tensor represents curvature of spacetime

$$G_{\mu\nu} = 8\pi \underset{\text{constant}}{K} T_{\mu\nu}$$

"Stress-energy tensor" represents energy, momentum & stresses

Important since stresses generate gravitational effects!

(1915)

But not enough wiggle room. So Einstein (1917) added a term

$$G_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi K T_{\mu\nu}$$

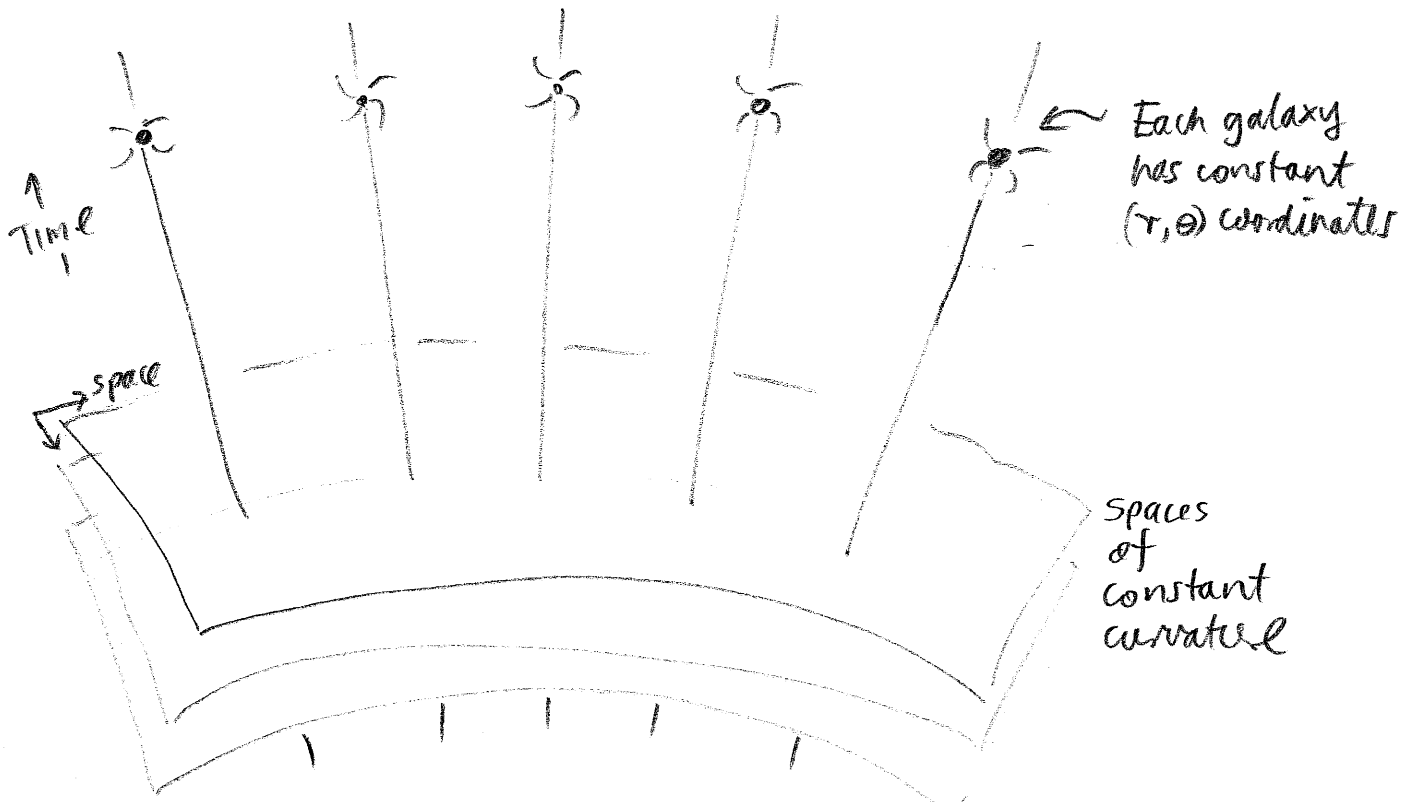
undetermined "cosmological constant"

Sign conventions from Misner, Thorne & Wheeler.	Henceforth set $K = 1$, $c = 1$

Robertson-Walker spacetimes = simplest relativistic cosmologies

← General cosmology with homogeneous, isotropic spaces

!! used in 97.4% of the cosmology literature !!



Line element $$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right]$$

$\underbrace{\text{Functional dependence of } R \text{ on } t = \text{complete dynamical history of the universe}}$

Finding this is 73.6% of the work in cosmology

e.g. R grows with t

⇓

galaxies get further apart

⇓

universe expands

Recover dynamics for R:

$$\boxed{\begin{array}{l}\text{Einstein field equations}\\ G_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi\, T_{\mu\nu}\end{array}}$$

specialize to case of

* Robertson-Walker spacetime
* Spatially homog., isotropic matter distribution

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p\, g_{\mu\nu}$$

ρ: Rest energy density of smoothed out matter

p: pressure of matter

u_μ: Four velocity of matter

$$\boxed{\begin{array}{l}\text{Basic equations of standard cosmological model:}\\[1em] \dfrac{\ddot{R}}{R} = -\dfrac{4\pi}{3}(\rho + 3p) + \dfrac{\Lambda}{3}\\[1em] \left(\dfrac{\dot{R}}{R}\right)^2 = -\dfrac{k}{R^2} + \dfrac{8\pi}{3}\rho + \dfrac{\Lambda}{3}\end{array}}$$

"$\bullet$" $= \dfrac{d}{dt}$

much easier than you thought!!!

Matter couples into cosmic dynamics merely by fixing values for ρ, p.

simple cases:

(I) Very slow moving dust (galaxies, particles of non-zero rest mass: protons, neutrons)

$\rho =$ rest energy density

$p = 0$

(II) Radiation (cosmic microwaves, em fields, neutrinos etc.)

$$\underbrace{\overset{\text{energy density}}{\rho} = 3\overset{\text{radiation pressure}}{p}}_{\text{Equivalent to assertion "Radiation has zero rest mass"}}$$

Read off dynamics: Case of $\Lambda = 0$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}\underbrace{(\rho+3p)}_{\rho+3p\,>\,0}$$

$\therefore\ \ddot{R} \overset{\text{always}}{<} 0$

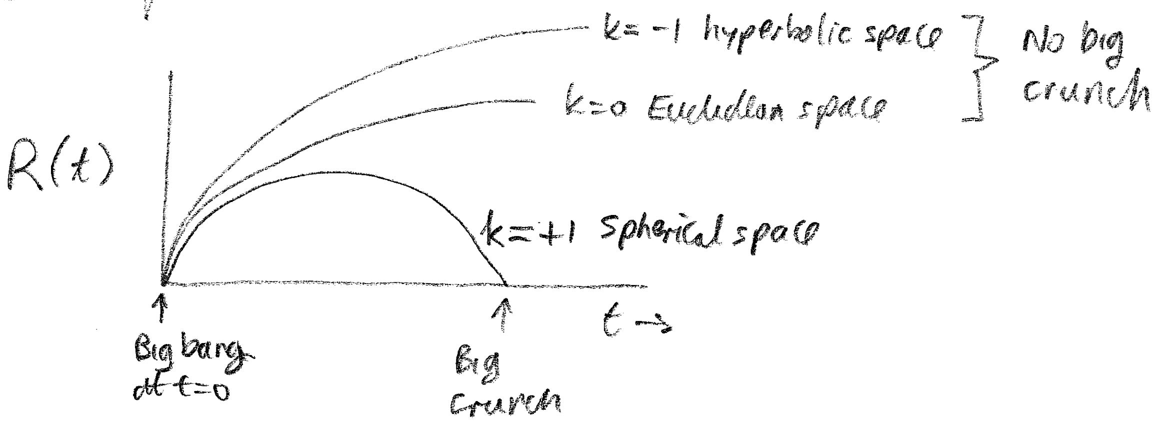
$\therefore$ Galaxies always accelerating towards eachother

$\boxed{\text{No static cosmology}}$

$$\underbrace{\left(\frac{\dot{R}}{R}\right)^2 = -\frac{k}{R^2} + \frac{8\pi\rho}{3}}$$

$k>0$: $\dot{R}$ can drop to zero. Cosmic expansion can halt & recollapse begin

$k \leq 0$ $\dot{R}$ cannot change sign. Cosmic expansion cannot halt



Case of $\Lambda \neq 0$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho+3p) + \frac{\Lambda}{3} \implies \boxed{\text{static cosmologies possible}}$$

Exerts repulsive "force" "negative pressure" Causes galaxies to accelerate away from each other

$$\ddot{R} = 0 \implies \Lambda = 4\pi(\rho+3p)$$

Table 5.1

DUST AND RADIATION FILLED ROBERTSON-WALKER COSMOLOGIES

case of $\Lambda = 0$

Spatial Geometry	Type of Matter: "Dust" $P = 0$	Type of Matter: Radiation $P = \frac{1}{3}\rho$
3-sphere, $k = +1$	$a = \frac{1}{2}C(1 - \cos\eta)$ $\tau = \frac{1}{2}C(\eta - \sin\eta)$	$a = \sqrt{C'}[1 - (1 - \tau/\sqrt{C'})^2]^{1/2}$
Flat, $k = 0$	$a = (9C/4)^{1/3}\,\tau^{2/3}$	$a = (4C')^{1/4}\,\tau^{1/2}$
Hyperboloid, $k = -1$	$a = \frac{1}{2}C(\cosh\eta - 1)$ $\tau = \frac{1}{2}C(\sinh\eta - \eta)$	$a = \sqrt{C'}[(1 + \tau/\sqrt{C'})^2 - 1]^{1/2}$

Wald's $a \leftrightarrow R$ here
$\tau \leftrightarrow t$

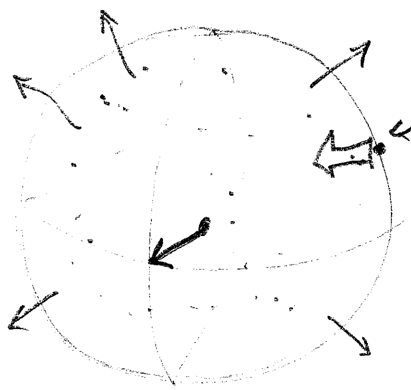
η is a parameter that couples a, τ

From R.M. Wald General Relativity

Visualize Dynamics via Newtonian Analogy

Milne, McCrea 1934

sphere of dust expands



sphere of radius $R(t)$

Expansion decelerated by inward pull of gravity according to Newton's inverse square law

measured by $\ddot{R}$

Recover **exactly** the equations of the relativistic model for R !!

Newton's inverse square law $\Longrightarrow$ $\ddot{R} = -\frac{4\pi}{3}\rho R$

Acceleration of unit mass on edge of sphere

$= -(\text{mass of sphere})/R^2$

$= -\frac{4\pi}{3}\rho R^3/R^2$

conservation of energy $\Longrightarrow$ $\underbrace{\frac{1}{2}\dot{R}^2}_{\text{kinetic energy}} \underbrace{-\frac{4\pi}{3}\rho R^2}_{\text{potential energy}} = \underbrace{-\frac{k}{2}}_{\text{constant}}$

$k = 0, -1$ $\longrightarrow$ Total energy non-negative $\longrightarrow$ cloud has sufficient kinetic energy to climb out of potential well $\longrightarrow$ no re-collapse

$k = +1$ $\longrightarrow$ Total energy negative $\longrightarrow$ cloud does not have sufficient energy to climb out of potential well $\longrightarrow$ re-collapse

Balance of radiation and ordinary matter

Early universe : Radiation dominated ⟶ Later universe : Ordinary matter dominated

$\rho_{rad} \gg \rho_m$

ρ_{rad}: energy density radiation

ρ_m: energy density dust

$\rho_{rad} \ll \rho_m$

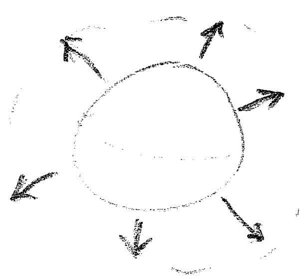
since

During expansion ρ_m dilutes as $1/R^3(t)$

ρ_{rad} dilutes a $1/R^4(t)$

Why?

Pick some volume $V(t)$ That moves with the cosmic expansion



In all geometries $V(t) \propto R^3(t)$

$\boxed{\rho_m}$ reduces solely because its energy is distributed over a greater volume

i.e. $\underbrace{\frac{d}{dt}(\rho_m V) = 0}_{\text{Energy conservation}} \Rightarrow \rho_m \sim \frac{1}{V} \sim 1/R^3$

$\boxed{\rho_{rad}}$ reduces because its energy is distributed over a greater volume AND its radiation pressure does lots of work during the expansion

i.e. Energy conservation: $0 = \underbrace{\frac{d}{dt}(\rho_{rad} V)}_{\text{dilution}} + p\underbrace{\frac{dV}{dt}}_{\text{pressure work}} = V\frac{d\rho_{rad}}{dt} + \frac{4}{3}\rho_{rad}\frac{dV}{dt}$

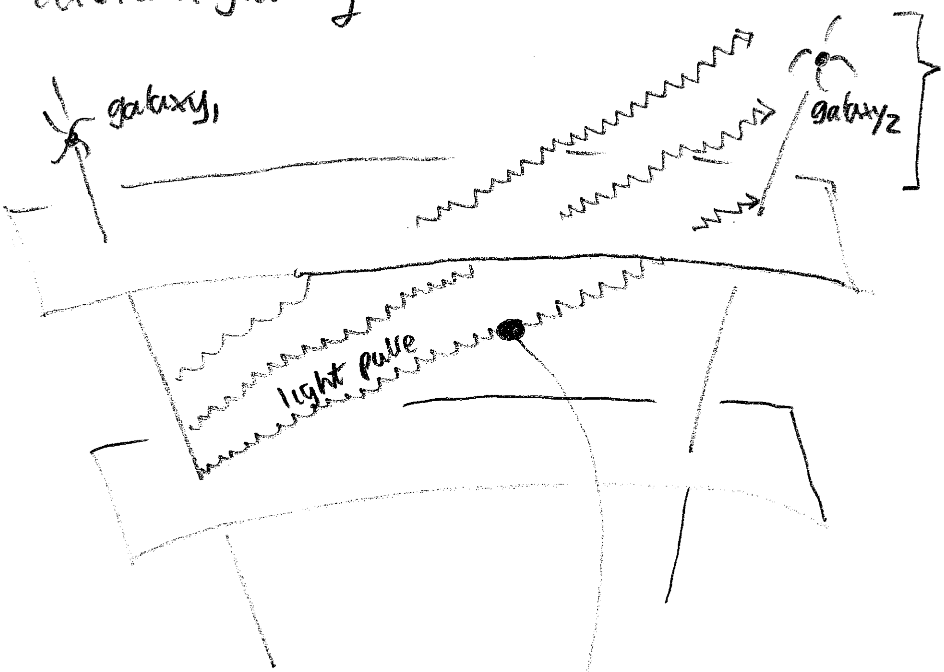
$p = \frac{1}{3}\rho_{rad}$

solve: $\rho_{rad} \propto V^{-\frac{4}{3}} \propto 1/R^4$

Observational foundations of modern cosmology (two of them!)

(I) Hubble red shift

Red shift in light from distant galaxy $\propto$ Distance to galaxy



Expansion of space
⇓
successive pulses of light from galaxy1 must travel further to reach galaxy2
⇓
greater delay in arrival later pulses
⇓
reduced frequency = red shift

(II) Cosmic microwave radiation at 2.7K

(Penzias + Wilson +)

↑

Diluted remnant of high radiation density in early universe

Photon

Energy = h . frequency

this reduces because of Hubble red shift (due in turn to cosmic expansion)

∴ this reduces too!

Reduction corresponds to "pressure work" loss

Add in volume dilution of photons too!

Appendix to Cosmology001: Critical Density

one equation for dynamics of RW spacetime

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{-k}{R^2} + \frac{8\pi\rho}{3} = H^2$$

Hubble's constant^2 since distance L between neighboring galaxies grows as

$$\frac{dL}{dt} = \frac{d}{dt}\left(\underbrace{\frac{L}{R}}_{\text{constant since expansion follows } R(t)} \cdot R\right) = \frac{\dot{R}}{R}L = HL$$

$$\boxed{\frac{k}{R^2} = \frac{8\pi\rho}{3} - H^2 = H^2\left(\frac{8\pi}{3H^2}\rho - 1\right) = H^2\left(\frac{\rho}{\rho_{crit}} - 1\right) \quad \text{where } \rho_{crit} = \frac{3H^2}{8\pi}}$$

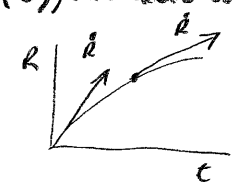
Read off directly:

$\rho = \rho_{crit}$	$\rho > \rho_{crit}$	$\rho < \rho_{crit}$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$k=0$	$k=1$	$k=-1$
Euclidean	Spherical	Hyperbolic

Flatness problem: During normal time (expansion) evolution, if $\rho/\rho_{crit} \neq 1$, ρ/ρ_{crit} moves to values ever further away from 1

from above $$\frac{\rho}{\rho_{crit}} = \frac{k}{H^2R^2} + 1 = 1 + \frac{k}{(\dot{R})^2}$$

$(\dot{R}(t))^2$ decreases with t in expansion



$\therefore \frac{1}{(\dot{R}(t))^2}$ increases with t during expansion

If at some time t

$(k=0)$	$(k=1)$	$(k=-1)$
$\frac{\rho}{\rho_{crit}} = 1$	$\frac{\rho}{\rho_{crit}} > 1$	$\frac{\rho}{\rho_{crit}} < 1$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$\frac{\rho}{\rho_{crit}}$ remains $=1$	$\frac{\rho}{\rho_{crit}}$ grows larger	$\frac{\rho}{\rho_{crit}}$ grows smaller

Exotic matter Needed to Solve Flatness Problem
(as is done in Inflationary Cosmology)
[$\lambda = 0$]

Normal matter:
$\rho + 3P > 0$

Exotic matter
$\rho + 3P < 0$

Einstein field equations
$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3P)$$

Normal matter:

$\ddot{R} < 0$

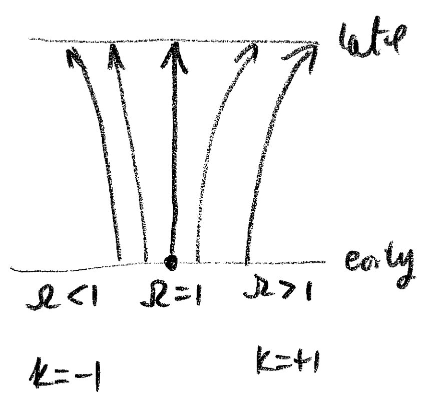
$\Downarrow$

$\dot{R} > 0$ but decreasing in expansion phase

$\Downarrow$

$$\Omega = \frac{\rho}{\rho_{critical}} = 1 + \frac{k}{\dot{R}^2}$$

DIVERGES from $\Omega = 1$ in expansion



Exotic matter:

$\ddot{R} > 0$

$\Downarrow$

$\dot{R} > 0$ and increasing in expansion phase

$\Downarrow$

$$\Omega = \frac{\rho}{\rho_{critical}} = 1 + \frac{k}{\dot{R}^2}$$

CONVERGES to $\Omega = 1$ in expansion